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Borders of disorder: in turbulent channel flow

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A quantitative theory of the average features of turbulent flow in a channel is described without the introduction of empirical parameters. The qualitative problem consists of maximizing the dissipation rate of the mean flow subject to the Rayleigh condition that the mean flow has no inflections. The quantitative features result from a boundary stability study which determines a smallest scale of motion in the transport of momentum. The velocity fields satisfying these conditions, the averaged equations and the boundary conditions uniquely determine an entire mean velocity profile at all Reynolds numbers within ten per cent of the data. The maximizing condition for the reproducibility of averages emerges from the Navier–Stokes equations as a consequence of a novel definition of nonlinear instability. The smallest scale of motion results from a theory for a time-dependent re-stabilization of the boundary layer following a disruptive instability. Computer reassessment of the several asymptotic estimates of the critical boundary eigenstructure can establish the limits of validity of the quantitative results.

1. Introduction

The non-periodic fluid motion called turbulence has attracted practical and theoretical attention for centuries. Distinct branches of this study include the average of flows in pipes and channels resulting from imposed pressure or flux, and separately the mixing resulting from mechanical stirring. This work addresses the phenomena in pipes and channels, both because of its practical importance and the many careful measurements of such flows. Although the observed flows are ever changing, for fixed forcing, the averages are remarkably reproducible (i.e. 'statistically stable'). Figure 1 shows mean profiles of velocity from recent measurements of pipe flow, by Zagarola, Perry & Smits (1997), over a large range of Reynolds numbers $\equiv R$ (the imposed quantity determining the flow). The logarithmic scale of the abscissa emphasizes the boundary region. The maximum of each profile marks the centre of that flow. The observation of profile slope, unchanging as R is increased, has been the subject of various theories for many years. This paper contains a quantitative theory for such profiles across the entire flow and for all R.

The central mathematical problem addressed here has only three elements. These are to find the maximum value of the dissipation rate of the mean flow as a function of R, subject to the condition that the mean flow has no inflections and that a spectral representation of the macroscopic momentum transport is limited by a smallest scale of motion.

It will be shown here in what sense this qualitative problem was explored previously by Malkus & Smith (1989, hereinafter referred to as MS). Earlier work anticipated aspects of the structure of the mean flow (Malkus 1979) and aspects of the nonlinear condition for reproducibility of an average (Malkus 1996*a*).

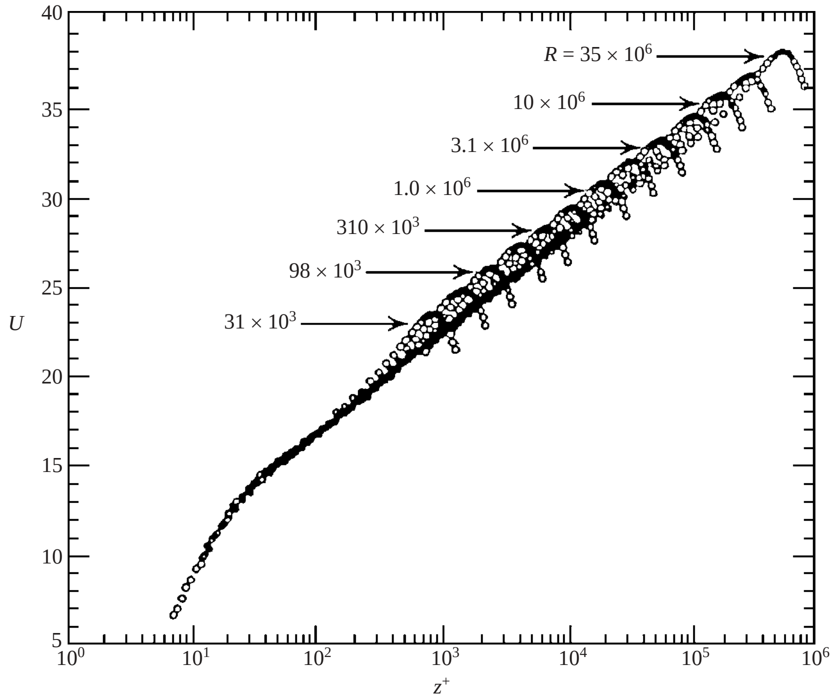

FIGURE 1. Plot of the observed velocity profiles in a pipe normalized using scaled variables, U for the velocity and z^+ for distance from the boundary (Zagarola *et al.* 1997).

In § 2, the Navier–Stokes equations and first integrals appropriate for incompressible channel flow are exhibited.

In § 3, of this paper the determination of a sufficient condition for instability of solutions of the Navier–Stokes equations is sought. A resolution of this problem is based on an unusual criterion for initial instability which leads to the conclusion that, among the solutions at a given R, only those with maximum dissipation rate of their mean flows can lead to reproducibility of their averages. This result is the first of the three premises of the formal problem, whose speculative origin is to be explored by comparison with the observations.

In § 4, the primary quantitative restriction is found in the investigation of a boundary-layer process which determines both the smallest scale of motion active in the momentum transfer process and a viscous spectral tail. This work is a generalization of papers by Howard (1964), and Malkus (1963, 2001) and determines the instability of a boundary layer re-establishing itself viscously after a violent disruption. A first step in this time-dependent problem leads to the classic Orr–Sommerfeld equation. Taylor (1923) was among the first to find and exhibit an instability of this equation (which is a two-dimensional vorticity equation central to this study). Before Taylor's quantitative study, Rayleigh (1880) deduced that steady parallel inviscid shear flow was unstable if the variation of the flow with position had an inflection. Raleigh's theoretical finding and its generalization to finite amplitude by Arnol'd (1965) is the basis for the second of the three premises of the formal problem. That there is a smallest scale of motion in the momentum transfer is the third premise.

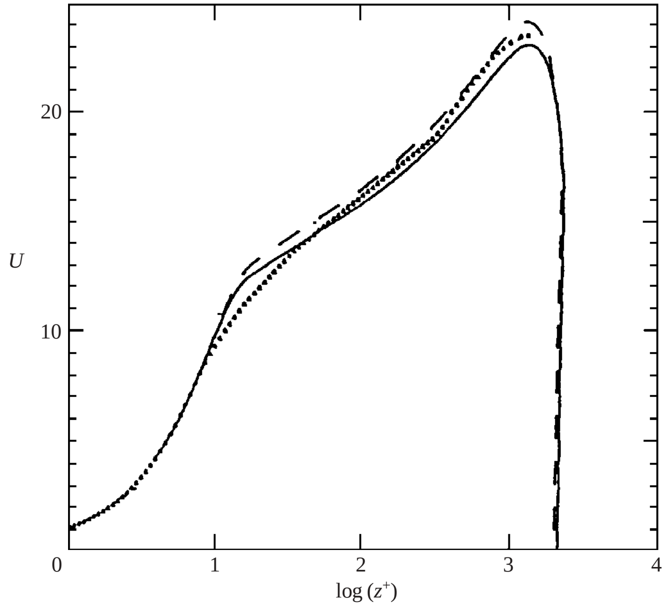

FIGURE 2. Recent experimental data for channel flow, and theoretical profiles for different values R_c of the boundary critical eigenvalue estimated in § 4. For each curve, $R \simeq 25\,600$; data . . ., Johanssen & Alfredsson (1983); theory ——, $R_c = 420$; — — —, $R_c = 480$.

In § 5, these quite different 'stability' conditions are shown to lead to a problem addressed at length in the empirical study (MS). Figure 2 from that study, using the quantitative results from § 4, is an example at a given R comparing the theory and data (but without the viscous tail which can smooth the transition from viscous to inertial regions). The inertial region exhibits a logarithmic slope that extends further into the flow at higher Reynolds number. The slope has a constant value as R is increased and the outer flow exhibits a velocity defect 'law'.

The number of contributions to the observation and interpretation of pipe and channel flow fill book after book. Relevant references to this vast work will be made here, for example the quantitative study of Goldshtik, Zametalin & Shtern (1982) explores the transition from the viscous profile near the boundary to the start of the logarithmic region just beyond, employing three assumptions drawn from the observations. Yet in the last hundred years there has been no deductive study free of empiricism which generates quantitative results for turbulent averages across the entire channel.

However, a formal determination of limits on turbulent flow has produced the 'upper bound' theory which was first successfully addressed by Howard (1963) then by Busse (1969), and recently by Doering & Constantin (1994), Nicodemus, Grossmann & Holthaus (1999) and Kerswell (1998) among others. Bounds on the total dissipation rate are determined, constrained by the power integrals, the boundary conditions, the mean momentum balance and the continuity condition. These formal bounds lead to plausible scaling laws for momentum transport. Yet an unrealistic aspect of all these bounds is their velocity fields, which if they actually occurred would be unstable. Kerswell (2002), using the same constraints, has been able to extend these studies to bounds on many other dissipation rate functionals. One of the functionals, the dissipation rate of the mean velocity (the functional studied here) does not have an unstable field. However, not one of the upper bound fields with the constraints listed above exhibit the logarithmic or velocity defect regions characteristic of the observed flows. All these limits are an order of magnitude larger than the observations. From the study to follow, it is anticipated that vorticity stability conditions imposed as additional constraints can bring these formal upper bound results much closer to the observations.

2. The Navier–Stokes equations and their first integrals

The equations for incompressible shearing flow in a channel are written

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} + \nabla P = \nu\nabla^2\boldsymbol{u} \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{1}$$

where $\boldsymbol{u} = U(z)\hat{\boldsymbol{i}} + \boldsymbol{v}$, and $\boldsymbol{v} = u\hat{\boldsymbol{i}} + v\hat{\boldsymbol{j}} + w\hat{\boldsymbol{k}}$ is of zero average, $\nu \equiv \mu/\rho$, is the kinematic viscosity, $\langle \partial P/\partial x\rangle$ and ρ are constants. The equations depend on the single parameter, the Reynolds number $R = \langle u\rangle d/\nu$, based on the z average of $U(z) \equiv \langle u\rangle$ and the channel half-width d. The brackets $\langle\rangle$ mean an average over the entire fluid. When scaled by d, the cross-space is z, $-1 \leqslant z \leqslant +1$.

The custom in much of the fluid dynamic literature is to scale equation (1) based on the boundary stress. Then for the steady averaged flow, the first integral of (1) is written

$$m + R_\tau^{-1}\beta = z, \tag{2}$$

which states that the average stress varies as z across the channel, where the 'Reynolds stress' $m \equiv \overline{uw}$ (the overbar indicates an average at fixed z), $\beta \equiv -\partial U/\partial z$, $R_\tau = U_\tau d/\nu$, $U_\tau \equiv \sqrt{\tau(-1)/\rho}$ called the 'friction velocity', $\tau(-1)$ is the stress at the boundary, also $z^+ \equiv (1+z)R_\tau$. Here and hereinafter all velocities are scaled with U_τ. Note that either R defined after (1) or R_τ defined after (2) can be fixed in an experiment. Then, the unfixed parameter is determined by the resulting flow. In this average steady state, integration of

$$\langle \beta z\rangle = RR_\tau^{-1} \qquad \text{and from (2),} \qquad \langle mz\rangle + RR_\tau^{-2} = \tfrac{1}{3}, \tag{3}$$

also from (2), the total dissipation rate, $D = \tau(-1)\langle u\rangle U_\tau$, is proportional to

$$RR_\tau^2 = R_\tau^2(\langle\beta^2\rangle + R_\tau\langle\beta m\rangle) = R_\tau^2(\langle\beta^2\rangle(1+I)), \tag{4}$$

where the non-dimensional

$$I = \frac{R_\tau\langle\beta m\rangle}{\langle\beta^2\rangle} = \frac{D_f}{D_m}. \tag{5}$$

D_m is the dissipation rate due to the mean flow and D_f is the dissipation rate due to the fluctuations. Also from (4),

$$I = \frac{R}{\langle\beta^2\rangle} - 1. \tag{6}$$

3. Proposed criteria for the instability and reproducibility of fully developed flow

An application of a formalism proposed in Malkus (1996*b*) is to determine a sufficient condition for the instability of a solution in fully developed steady-state channel flow. A disturbance equation, not linearized, for some solution $\boldsymbol{v}_0$, $U_0(z)$, P_0

from (1) is written for $\boldsymbol{v}'$ and p',

$$\partial \boldsymbol{v}'/\partial t = -\nabla p' + R_{\tau_0}^{-1}\nabla^2 \boldsymbol{v}' - w'(\partial U_0/\partial z)\hat{\boldsymbol{i}} + \boldsymbol{v}'\cdot\nabla\boldsymbol{v}_0 + \boldsymbol{v}_0\cdot\nabla\boldsymbol{v}' + \boldsymbol{v}'\cdot\nabla\boldsymbol{v}' \tag{7}$$

where $\langle \boldsymbol{v}' \rangle = 0$. The instability of a fully developed flow is established by any example of a disturbance for which the mean squared value of the resulting fluctuating fields grows with time. If growth occurs in an arbitrarily chosen first instant, t_0, for all possible phases and amplitude of the disturbance field, it is proposed that this is a sufficient condition to assure instability. The condition for growth of the mean squared value of $(\boldsymbol{v}_0 + \boldsymbol{v}')$ is that

$$\frac{\partial}{\partial t}(\langle 2\boldsymbol{v}_0\cdot\boldsymbol{v}'\rangle + \langle \boldsymbol{v}'^2\rangle) > 0. \tag{8}$$

This condition also must be met by that subset of disturbances of arbitrary amplitude, which over the entire horizontal domain are not correlated with $\boldsymbol{v}_0$ at $t = t_0$ (e.g. other turbulent flows and periodic cellular solutions). For these selected disturbances, $\langle \boldsymbol{v}_0\cdot\boldsymbol{v}'\rangle$ vanishes from the condition for instability (8), as does $\langle \boldsymbol{v}'\cdot\boldsymbol{v}'\cdot\nabla\boldsymbol{v}_0\rangle$ from $\langle \boldsymbol{v}'\cdot(7)\rangle$ at $t = t_0$. Using the average integral of $\boldsymbol{v}'\cdot(7)$, then (8) is written fully as

$$\frac{\partial\langle \boldsymbol{v}'^2\rangle}{\partial t} = \left\langle R_{\tau_0}^{-1}\boldsymbol{v}'\cdot\nabla^2\boldsymbol{v}'\right\rangle + \langle m'\beta_0\rangle > 0. \tag{9}$$

This test of the instability of the solution $\boldsymbol{v}_0$, U_0, P_0, is to be made with disturbances $\boldsymbol{v}'$, which have the form (but arbitrary amplitude) of another solution (typically unstable). If then m' is proportional to a solution m_1, from equation (2), $m_1 + R_{\tau_1}^{-1}\beta_1 = z$. Also from the basic equations and (2), the average fluctuation dissipation rate term is

$$-R_{\tau_1}^{-1}\langle \boldsymbol{v}_1\cdot\nabla^2\boldsymbol{v}_1\rangle = \langle m_1\beta_1\rangle = \langle z\beta_1\rangle - R_{\tau_1}^{-1}\left\langle \beta_1^2\right\rangle. \tag{10}$$

Since the disturbance amplitude of $\boldsymbol{v}'$ appears quadratically in (9) and $\boldsymbol{v}'$ is defined as proportional to $\boldsymbol{v}_1$, using (10), (9) may be rewritten

$$\frac{\partial\langle \boldsymbol{v}'^2\rangle}{\partial t} = c\left[R_0 - R_1 + \left\langle \beta_1^2\right\rangle - R_{\tau_0}R_{\tau_1}^{-1}\langle \beta_1\beta_0\rangle\right] \geqslant 0, \tag{11}$$

where c is an arbitrary positive constant. If then R is fixed, Schwartz's inequality establishes that a sufficient condition for instability of $\boldsymbol{v}_0$, U_0, P_0 is that

$$R_{\tau_0}^2\left\langle \beta_0^2\right\rangle < R_{\tau_1}^2\left\langle \beta_1^2\right\rangle. \tag{12}$$

The literature (e.g. Salmon 1988; Friedlander & Vishik 1992), contains many studies of sufficient conditions for the instability of steady non-viscous flows. In contrast, the average condition, equation (12), is for non-steady or steady flow and is derived for viscous flows (the average dissipation rate of a mean flow or entropy production rate of a mean flow in the Boussinesq sense).

Given the conclusion, equation (12), a necessary condition for reproducibility of averages for fixed R is that the subset of solutions for which $R_\tau^2\langle\beta^2\rangle$ is (very close to its) maximum will be selected.

To continue with this study, this novel condition may be taken as a premise, whose consequences will be explored in the following pages.

A first implication of maximum $R_\tau^2\langle\beta^2\rangle$, for fixed $R = R_\tau\langle\beta z\rangle$ from (3), can be investigated by seeking the maximum of the homogeneous functional

$$\frac{R_\tau^2\langle\beta^2\rangle}{R^2} = \frac{R_\tau^2\langle\beta^2\rangle}{R_\tau^2\langle\beta z\rangle^2} \equiv B = \frac{\langle\beta^2\rangle}{\langle\beta z\rangle^2}. \tag{13}$$

Using the Schwartz inequality, B is seen to have a minimum value of 3, but has no maximum without additional constraints on β. It is anticipated that as R increases, β will become larger in the boundary regions. The constraint of a smallest spatial scale, $\equiv \delta_\nu$, for β at a given R and the Rayleigh requirement that $\partial\beta/\partial z$ be of one sign for stability, are the topics of the next section. With these constraints, B maximum is achieved when $\langle \beta z\rangle^2 = R_\tau^2 0(\delta_\nu^2)$, while the value reachable by $\langle\beta^2\rangle$ is $R_\tau^2 0(\delta_\nu)$, recalling from equation (2) that the value of β^2 at the boundaries is R_τ^2 and that β is scaled by U_τ. Hence, $\langle\beta^2\rangle$ approaches a minimum value to maximize B. Then, from equation (6) the non-dimensional ratio I will be a maximum for the stable solutions. That the R_τ^2 resulting from a minimum $\langle\beta^2\rangle$ would lead to qualitatively plausible results for channel flow was 'discovered' in the empirical study (MS). The functional with that maximal property was labelled there an 'efficiency function', but with no knowledge of its origin. It will be reassessed here using the quantitative results reported on in the following section. The set of velocity fields from which an extreme will be selected is still to be determined.

4. A quantitative theory for the boundary layer in fully turbulent shear flow

Often called the 'laminar sublayer', this boundary flow is seen as the central amplitude-determining process in turbulent channel flow. An unusual view of convective instability due to Howard (1964) was applied to turbulent shear flow in Malkus (2001). Howard pictured the convective instability as sweeping away a large part of the thermal boundary layer, which would then 'heal' diffusively until once again unstable. The critical condition for instability was not computed, but assumed to be similar to that found on a time-independent thermal profile not influenced by any large-scale velocity fields. (Supported in a parallel quantitative study, Malkus 1963). Both assumptions appear sound for convection at Rayleigh numbers that are not too high, but are not appropriate at high Rayleigh number and for turbulent shear flow. In shear flow, the instability is quite sensitive to the time-changing shape of the mean flow profile, e.g. in the re-establishment of flow in channels just above the critical Reynolds number (Malkus 2002).

Here, also, the quantitative boundary Reynolds number is sought for initial instability of the diffusively healing layer. Such an estimate was made in Malkus (2001), yet a more detailed analysis of the unstable field of motion is required in order to determine the position of the transition from viscous to inertial behaviour. A complete study of this transient regrowth and instability will require an extensive computer exploration. Here, an idealization of this process will be analysed using several of the methods of estimation developed over the years for such boundary-layer problems.

The plausible development of the 'healing' diffusive layer is the studied and restudied Blasius-layer instability. An important process, akin to the first instability of flow over an aircraft wing, Blasius flow has been explored in many experiments as well as in theory. Although it is an instability on a growing mean flow profile, it has been treated successfully as an instability on a time-independent flow with corrections made for the non-parallel change in the profile. This basic Blasius self-similar profile, seen in figure 3, is derived and plotted in Schlichting (1960). The critical Reynolds number ($R=\delta V/\nu$, where δ is the 'displacement thickness' of the boundary layer, V the 'outer' velocity at 3δ and ν the kinematic viscosity) $R_{critical} \equiv R_c$ is found by W. Tollmein (1929) as $R_c = 420$, by C. C. Lin (1945) as $R_c = 421$ by R. Jordinson (1970) as $R_c = 520$, all reported at length in Drazin & Reid (1981). The curves of marginal stability for both

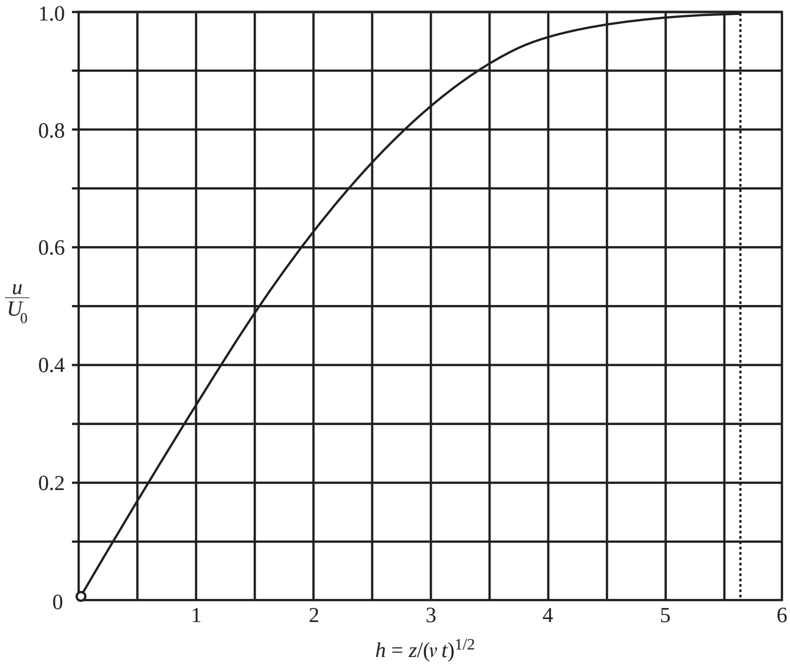

FIGURE 3. Scaled Blasius velocity distribution in the laminar layer as measured by Nikaradse, replotted in Schlichting (1960).

parallel and non-parallel flow are compared with experimental data in figure 4. The critical values reported seemed insensitive to 'noise' in the input flow, supporting the belief that the instabilities observed were supercritical. It may be fortuitous that both data and theory recapture the earliest uncorrected results.

The theory for this instability emerges from the Orr–Sommerfeld linear two-dimensional vorticity equation, written

$$(\mathrm{i}\alpha R)^{-1}(\mathrm{D}^2-\alpha^2)^2\phi = (U-c)(\mathrm{D}^2-\alpha^2)\phi - U''\phi, \tag{14}$$

with boundary conditions

$$\alpha\phi = \mathrm{D}\phi = 0 \quad \text{at } z=-1,\ z=+1,$$

where

$$\psi' = \phi(z)\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)} \tag{15}$$

is a streamfunction and $\nabla^2\psi'$ is the y-component of the vorticity. To determine the critical eigenstructure, U in equation (14) initially is taken to be a time-independent flow, as in the Blasius study. A first consequence of this stability equation is Rayleigh's deduction that, at high R, instability can occur if U'' changes sign. In consequence of the presumed spasmodic nature of the stabilizing processes, not only near the boundary but throughout the fluid, it is proposed that overall stability can only be assured if U'' is of one sign for the entire channel. However, at lower R, equation (14), which is of fourth order, has two 'outer' primarily inertial solutions and two 'inner' solutions which decay exponentially away from the boundary. The separation ratio of the critical point where $U=c$, which is at 0.725δ for $c=0.40$ in Blasius flow, and the inner layer, which determines the transition from partially inertial to purely viscous

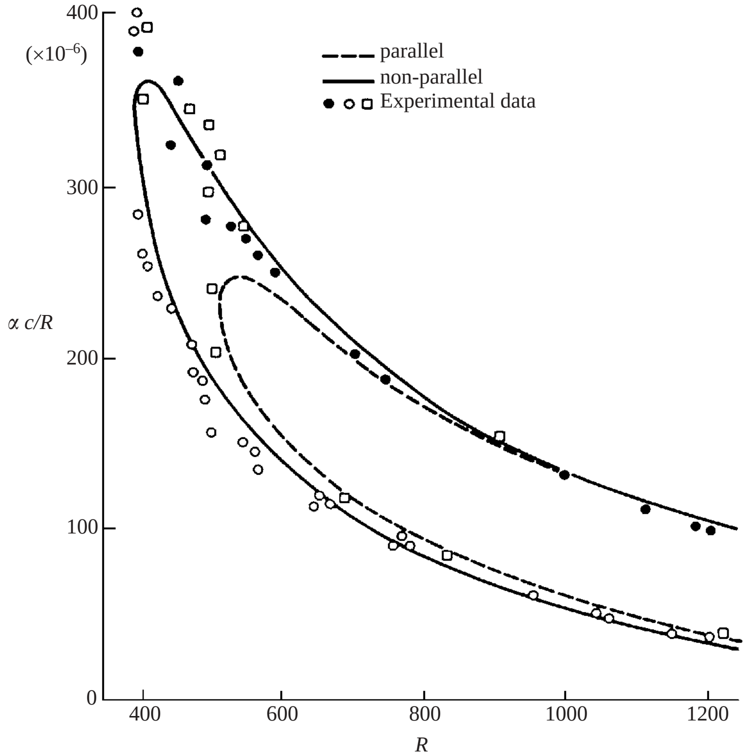

FIGURE 4. Curves of marginal stability for the Blasius boundary layer based on parallel and non-parallel stability theory and experimental data, from Drazin & Reid (1981).

behaviour is estimated in Drazin & Reid (1981) (pp. 166–172) as

$$\left[(\alpha R)^{1/3} + (\alpha R)^{1/5}\right] / 2(\alpha R)^{1/2} \simeq 0.307 \tag{16}$$

for $R_c = 420$ and $\alpha \simeq 0.30$, where for the 'healing' channel flow, $U''(-1)$ is not zero.

With a smallest scale of motion in the inertial process to be determined using equation (16) (followed by an exponential viscous 'tail') and the requirement that U'' be one sign for stability, a maximization of the I in equation (6) can be implemented. Presumably the minimization of $\langle \beta^2 \rangle$ and the following determination of R_τ^2, subject to these few constraints, will bound the realized flow from above. The comparison of the observed flow and the constrained optimization will then determine the importance (Virk 1975; Malkus 1979) or lack thereof, for additional constraints. The following section rephrases the central features of the earlier exploration (MS) which included the optimization of I and determination of R_τ^2 at fixed R, here using the derived eigenvalue R_c and the Blasius based estimate for the relation between R_τ and the smallest scale of motion.

5. An optimum channel flow

The constraint that in this flow U'' be of one sign is implemented with a Fejér representation

$$-U'' \equiv F^* F, \qquad F \equiv \sum_{0}^{\infty} F_k \, \mathrm{e}^{\mathrm{i}k\phi}, \tag{17}$$

where $\phi \equiv (1+z)\pi$ and F^* indicates the complex conjugate. Results with another representation, using product Chebyshev series due to Worthing (1990), will also be discussed. The boundary conditions, $\boldsymbol{v} = 0$, and $\nabla \cdot \boldsymbol{v} = 0$, require that

$$U''' = U = 0, \quad U'' = R_\tau, \quad U' = \pm R_\tau \text{ at } z = \mp 1. \tag{18}$$

The two constraints on the series F that follow from (18) are

$$\sum_0^\infty F_k = R_\tau^{1/2}, \quad \sum_0^\infty F_k^2 = R_\tau. \tag{19}$$

An average over the second integral of U'' in (17) leads to

$$R = \tfrac{1}{2} R_\tau^2 - R_\tau \sum_0^\infty \sum_{\substack{0\\k\neq j}}^\infty \frac{F_k F_j}{\pi^2 (k-j)^2}. \tag{20}$$

Lastly, that position in the boundary layer where the inertial behaviour ceases and the exponential viscous tail begins, defines a largest wavenumber, $\equiv k_\nu$, effective in momentum transport. From the estimate, equation (16), and the distance of the Blasius critical layer from the boundary 0.725δ, that smallest scale can be written $\delta_\nu = (0.307 \times 0.725)\delta$, recalling that δ was defined as the 'displacement thickness' of the boundary layer which determined the critical R_c ($=420$ for the Blasius profile). In the viscous boundary, a local Reynolds number can be defined as $R(s) = Vs/\nu$, where V and s are the unscaled average velocity and distance from the boundary. The scaled quantities in that region are $U = V/U_\tau = z^+$, $z^+ = (1+z)R_\tau$, and $(1+z) = s/d$, then

$$R(s) = z^{+2}, \quad (1+z) = R_\tau^{-1} R(s)^{1/2}. \tag{21}$$

The smallest scale of motion δ_ν determines a full wavelength of largest wavenumber $k_\nu = 2d/\delta_\nu$ in the Fejér series, (17). Therefore, from (21),

$$k_\nu = \frac{2R_\tau}{[R(\delta_\nu)]^{1/2}} = \frac{2R_\tau}{(0.307 \times 0.725 R_c)^{1/2}} = 0.207 R_\tau \tag{22}$$

for $R_c = 420$.

Hence, the infinite series in equations (17)–(20) can be truncated at k_ν, if it is presumed that the summable exponential tail has a small effect which can be added later on. (Such a summation is made in the 1979 paper.)

To maximize I of equation (6), the minimum value of $\langle \beta^2 \rangle$ as a function of R is to be found. The earlier paper (MS) contains both the structure of $\langle \beta^2 \rangle$ and the Lagrangian for minimum $\langle \beta^2 \rangle$ subject to the boundary conditions, a smallest scale k_ν and the Reynolds number given by equation (20). Briefly outlining that work,

$$\langle \beta^2 \rangle = R + \frac{1}{\pi^2} \sum_0^{k_\nu} \sum_0^{k_\nu} \sum_0^{k_\nu} \sum_{\substack{0\\k\neq j}}^{k_\nu} \frac{F_k F_j F_m F_n}{(k-j)^2} \delta_{k-j,m-n} - \tfrac{2}{3} R_\tau^2, \tag{23}$$

where $\delta_{k-j,n-m}$ is the Kronecker-delta function. The Lagrangian for minimum $\langle \beta^2 \rangle$, subject to the constraints equations (19), (20) and (22) lead to a set of $[k_\nu + 4]$ equations for $F_k : k = 0, k_\nu$ and $\lambda_1, \lambda_2, \lambda_3$, which are

$$\frac{(\lambda_3 - 1)}{2\pi^2} R_c^{1/2} k_\nu \sum_{\substack{0\\k\neq j}}^{k_\nu} \frac{F_j}{(k-j)^2} + \lambda_1 + 2\lambda_2 F_k + \frac{4}{\pi^2} \sum_0^{k_\nu} \sum_0^{k_\nu} \sum_{\substack{0\\k\neq j}}^{k_\nu} \frac{F_j F_m F_n}{(k-j)^2} \delta_{k-j,n-m} = 0, \tag{24}$$

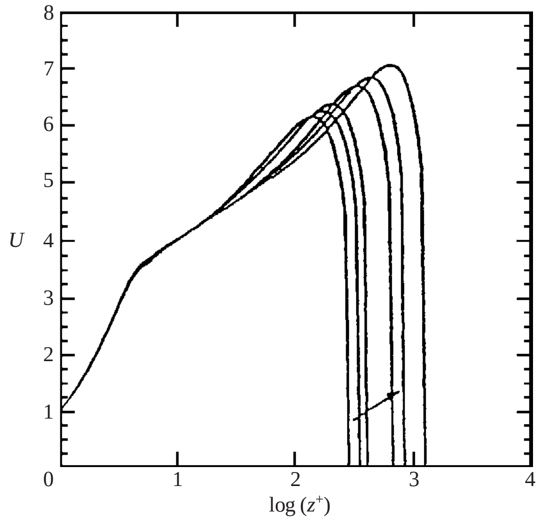

FIGURE 5. Maximum profiles for $k_\nu = 100, 122, 144, 236, 303$ and 450.

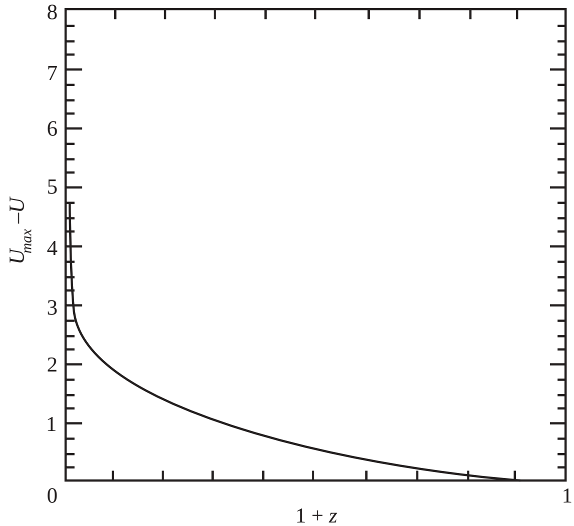

FIGURE 6. Maximum velocity defect profiles for the six curves of figure 5. The exact superposition suggests the title 'law'.

and was solved using Newton's method. With these solutions for $F_k(k_\nu)$, the value of R_τ^2 is found from equations (22) and (20) for a given Reynolds' number. As a function of Reynolds number, the mean flow for this optimization is shown in figures 5, 6 and 7, reproduced from MS, but with the numerical results found here. An earlier study, Malkus (1979), anticipated qualitative consequences of an F_k spectrum such as figure 7. There, the 'smoothness' of a spectrum, with the exception of F_0 and F_1 as in figure 7(*b*), permits the determination of $U(\varphi)$ for $\varphi \gg k_\nu^{-1}$ as

$$U_{MAX} - U = \frac{1}{\pi^2}\left[F_1^2 \ln \operatorname{cosec} \frac{\varphi}{2} + \tfrac{1}{2} F_0(F_0 - F_1)(\pi - \varphi)^2\right] + 0\left(\frac{F_2}{k_\nu}\right), \qquad (25)$$

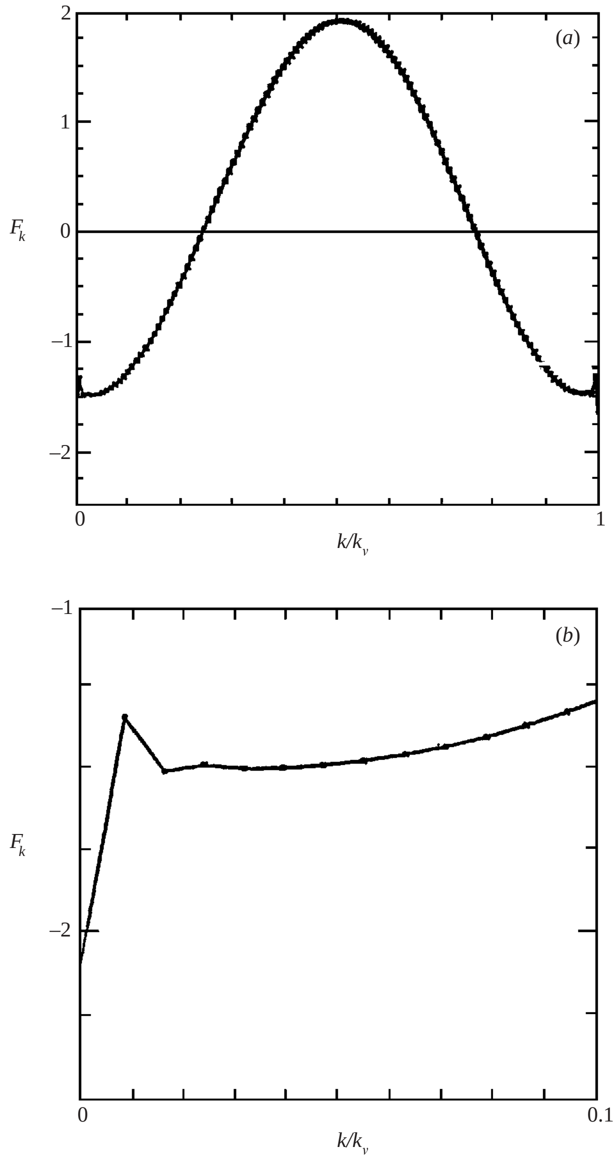

FIGURE 7. (*a*) Maximum F_k spectrum for $k_\nu = 144$. (*b*) Close-up of the spectrum 'boundary layer'.

which is equation (5.2) in Malkus (1979). It is seen that the unsmooth initial terms in F_k determine the internal flow. In MS, the R dependence of F_0 and F_1 for minimum $\langle \beta^2 \rangle$ vanished with increasing R. This is shown in figures 5 and 6, which are integrals for $U(\varphi)$ computed from the optimum F_k spectrum. The intercept of the smooth F_k spectrum as k approaches zero, determines the logarithmic slope, remaining constant only if that intercept is independent of R, as in figure 5. The velocity defect 'law' results from the R independence of the low k values of F_k, as exhibited in figure 6. All other moments of β, m and their products can be determined from the $F_k(R)$ and equation (2) (e.g. the average fluctuation dissipation rate as a function of position and R).

Figure 2 compares theoretical curves for $R_c = 420$ and 480 with data at $R = 25\,600$. For Reynolds numbers greater than 50 000, the computer time required to determine the optimum spectrum became excessive. Encouraged by Professor Ierley at Michigan Technical University, Worthing (1990) repeated the qualitative aspects of the problem in MS, but used a Chebyshev series permitting both a check on the work in MS and results at much larger R. Perhaps his approach will be repeated with a reassessment of the quantitative determinations reported in this paper.

The exploratory qualitative study in MS included a search of consequences of maximizing integrals other than I of equation (6). None were found that even approached the observed flow. In contrast to the maximum of the dissipation rate of the mean, the maximum of the total dissipation rate, equation (4), was determined in MS from an Euler–Lagrange equation which was linear in F_k, hence more easily solved. The F_k for k near zero for this case approached zero as R increased, leading to a velocity profile flat in the interior with an abrupt boundary transition.

The necessary condition for reproducibility found here appears to be closely approached in channel flow without additional constraints. It is probable that this is due to a dense occupancy of unstable solutions in the phase space and may not occur near some initial instabilities. A first qualitative formulation of this condition for flows where there is a variation of density with position (e.g. convection) was made in Malkus (1996*a*). Studies for reproducibility of averages in flows with magnetic fields (e.g. dynamos and plasmas), Malkus (1996*b*), have been explored only in limited parameter ranges. Application of the approach taken in this paper to those adjacent problems can lead to quantitative results, perhaps as here, close to the observed averages.

6. Continuing exploration of the boundary flow

The apparent success of the Blasius idealization and of the estimates of boundary eigenstructure from Drazin & Reid may be fortuitous. A computer study of the diffusive regrowth of the boundary region with a variety of 'wounds' due to instabilities and the point of recurring instability as they heal, can refine the quantitative estimates made here. The average of the realized disruptions will provide a better picture of the observed 'laminar boundary layer' (including apparent 'fluctuations' due to averages over various phases of the diffusive recovery, as Howard (1964) explored in his thermal convection study). Also needed is computer assessment of the spatial starting point of the inertial aspect of the eigenflow which establishes the relation between k_ν and U_τ (i.e. equation (22)). None of these suggestions involve programming to deal with aperiodic flow, rather they will address further exploration of the proposed laminar time-dependent stability constraint.

7. Conclusions

Two unusual 'stability' conditions are advanced to predict detailed quantitative observations of the averages in turbulent channel flow. The first condition is that, for reproducibility of the average flow at a given Reynolds number, the observed solutions of the Navier–Stokes equation will be those solutions whose mean flows have the maximum dissipation rate. The second condition results from the presumption that recurring instability removes part of the boundary layer leading to the time-dependent problem of a diffusively 'healing' viscous boundary. This requires that, for stability, the mean flow has no inflections. A central condition determined in this stability problem is the smallest spatial scale of momentum transport. This determination fixes the quantitative features of the theory, and can be refined with further computer study. The qualitative aspects of this theory determine detailed features of the profile, such as the logarithmic region and the velocity defect 'law', across the entire flow and at all Reynolds numbers.

Establishing what turbulence does perhaps can help us to understand what turbulence is. The view underlying this study is that reproducible observations of the flow are averages of the least unstable of the many unstable solutions of the basic equations. Yet more is learned each year about this disordered flow by those peering in with computers from the borders of disorder described here.

Such novel results as this study contains finally depended upon the demanding computations performed by Professor Leslie Smith to determine the extreme F_k. Following that herculean effort, she applied our theoretical framework to the Couette problem of flow between parallel plates in relative motion (Smith 1988, 1991). The symmetry of that problem, of the modified Rayleigh condition (due to Fjørtoft) and the boundary stability problem, all differ from the parallel flow problem treated here. It should be of value to test the generalizability of the quantitative work in this paper in that allied problem.

Suggestions for improvement in this presentation by the referees and by senior participants in the GFD 2002 summer program on 'upper bounds' were much appreciated.

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